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by

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# Stable and Ground States of Dipolic

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**INTRODUCTION** Up to now there is no exact description of ground and stable states of classical spins lattice system with dipolar long range interaction. Its necessity is connected with several aspects. On the one hand, lattice dipole systems draw theoreticians' attention not only by the possibility of application of new physical ideas but nontrivial results got on that models. And the reality of dipolic interactions would always be attractive [1-8, 11, 20-22, 30-32]. On the other hand, one can observe dipolic forces in many physical phenomena beginning with ferromagnetic disintegration into domains up to proteins and lyotropic liquid crystal systems self-organization [8, 10, 11, 21, 25-27].

The development of mathematical methods [12-16] and obtaining of new analytical and numeric results on the ground state lattice systems with dipole interaction [18,19] allowed to determine precise structure of ground and metastable states. Further we analyse ground state problems limiting ourselves to classical spins dipole interactions. We pay special attention to strict formulations of all restrictions and conditions as in the problem under study we can't always find nonobvious considerations that can affect the results.

Dipolic *is a new term that means abstract substance built of spheres with finite size and in the middle of these spheres there are point ( with linear size much less than the sphere radius) dipole moments.* From physical point of view dipolic is condensed state of Stockmayer liquids [30, 31]. All physical properties of dipolic i.e. ground state, metastable states, inner motions and phase transitions are conditioned only by dipole-dipole interaction. The importance of the dipolic model is based upon its stability, numerous phases and the possibility of complete theoretical analyses. Besides dipolic serves

as an excellent model of many real phenomena. The bright example of it is in polystyrene spheres ordering on water surface, micro structuring of ferromagnetic particles covered by polymers in liquid, amphiphilic molecules polar heads orientation in micelles, on water-oil interfaces and other much more complex phenomena in heterodispersed phases, determined by dipole interaction forces.
We consider the following aspects. In Section 1 we analyze the problem of existing of ground state in lattices with arbitrary dimensionality. Subsection 2.1 contains the analyses of two dimensional systems with three dimensional interaction. Worked out methods of analyses are applied in Subsections 2.3-2.5 for study of rhombus systems. In Subsection 3.1 degeneracy space is defined and a dipole system order of parameter on a simple cubic lattice is built. In Subsection 3.2 we proceed topological study of cubic dipolic stable inhomogeneous configurations without discontinuities in the order of parameter field.

# 1 Stable configurations of dipole systems

For analyses of stable configurations of dipolic with arbitrary dimension we consider the system of $N$ classical point dipoles $P_i^\alpha$ localized in the points $r_i^\alpha$ of $\mathbf{R}^n$ space. Let $r_{ij}^\alpha = r_i^\alpha - r_j^\alpha$, if $i \neq j$. Subscripts $i, j, k, l, m = 1, 2, ..., N$ denote a dipole, and superscripts $\alpha, \beta, \gamma = 1, 2, ..., n$ are used to denote the components of $n$-dimensional orientation vectors $\vec{P}_i$, and the coordinates $\vec{r}_i$ . We set the energy of dipole - dipole interaction as

$$H = \frac{1}{2}\sum_{ij}\sum_{\alpha\beta} A_{ij}^{\alpha\beta} P_i^\alpha P_j^\beta \tag{1}$$

were the dipole tensor of $R^n$

$$A_{ij}^{\alpha\beta} = \frac{\partial^2}{\partial r_i^\alpha \partial r_j^\beta}(|r_{ij}|^{2-n}) = |r_{ij}|^{-n}(\delta_{\alpha\beta} - n r_{ij}^\alpha r_{ij}^\beta / |r_{ij}|) \tag{2}$$

and distance $R^n$ is commonly set as

$$|r_{ij}| = \sqrt{\sum_\alpha (r_{ij}^\alpha)^2} \tag{3}$$

Consider that $A_{ij} = 0$, if $i = j$. We call configuration to be a set of vectors

$$\mathcal{R} = \{\vec{P}_1, \vec{P}_2, ..., \vec{P}_N\} \tag{4}$$

Sometimes we will call the the configuration (4) as orientational or dipole structure in contrast to the spatial dipolic structure defined by a set of coordinates

$$\mathcal{G} = \{.\vec{r}_1, \vec{r}_2, ..., \vec{r}_3\} \tag{5}$$

and Hamiltonian (1). Let's call such configuration to be $\mathcal{R}$, where there is minimal energy of dipole interaction. It's naturally to count $\mathcal{R}^0$ configuration - that corresponds to the global minimum of dipole energy (1) - as the ground state. Locating of $\mathcal{G}$ - structure is significant, otherwise the approaching of point dipoles may be destroyed as a result of their convergence.

The main problem is clear: to find the ground state $\mathcal{R}^0$ of a dipole system with Hamiltonian (1). Using Tikhonov theorem about direct product compactness and Weierstrasse theorem about the possibility to reach the minimum we assume that every finite system with dipole interaction has its own ground state. the method of finding of stable dipolic configuration is easily seen from (1). The energy $H$ is a quadratic form by $P_i^\alpha$ variables. Thus, the solution of the extreme task is to find eigenvalues and eigenvalue $\lambda_0$ will correspond to $\mathcal{R}^0$ ground state configuration.

It is not so easy to solve the problem of the ground state in infinite ($N \to \infty$) systems. The main problem is connected with the arbitrary convergence of $n$-dimensional lattice sums [4, 5].

$$\Phi_i^{\alpha\beta} = -\sum_j \left( \frac{\delta_{\alpha\beta}}{|r_{ij}|^n} - n\frac{r_{ij}^\alpha r_{ij}^\beta}{|r_{ij}|^{n+2}} \right) \tag{6}$$

that take the place, for instance, in calculating the field on $i$ site

$$h_i^\alpha = -\sum_j \sum_\beta A_{ij}^{\alpha\beta} P_j^\beta \tag{7}$$

if $\vec{P}_j$ equals a constant vector $\vec{P}_0$ (Ferromagnetic configuration). Notice, that local fields (7) are naturally introduced as $\vec{h}_i = -(\partial H/\partial \vec{P}_i)$ to present energy

(1) in the form

$$H = -\frac{1}{2}\sum_i \sum_\alpha h_i^\alpha P_i^\alpha. \tag{8}$$

It is postulated in [17], that the dipole interaction potential (2) is not fully integrated, but its sphere mean equals zero. The latter case is connected with the local nonintegrability of every term in (6) in the vicinity of every $i$ site, though the sphere mean zeroing leads to some harmonicity. There we see certain dualism: if the series absolutely converges (6), then the problem of an absolute series convergence (1) arises. Therefore, the ground state depends on the method of summation and, hence, on the form of the sample.

# 2 Analyses of two-dimensional systems with three–dimensional fields

We assume the following consideration: the ground state of any system with classical spins - arranged in a plane and interacting according the law (1) - has the configuration with spin orientation being set in the same plane.

## 2.1 Stability conditions for two-dimensional systems

To describe plane (two-dimensional) dipole configuration we introduce a complex variables:

$$P^\alpha = \zeta, \; z = r^\alpha, \; z_{ij} = z_i - z_j = r_{ij}^\alpha, \; [\zeta_a\bar{\zeta}_b + \bar{\zeta}_a\zeta_b] = 2(.\vec{a}\cdot\vec{b}). \tag{9}$$

Substituting (9) in (1) and (7) we find the energy

$$H = -\frac{1}{8}\sum_{i,j}\left(\frac{\zeta_i\bar{\zeta}_j + \bar{\zeta}_i\zeta_j}{|z_{ij}|^3} + 3\frac{\zeta_i\zeta_j\bar{z}_{ij}^2 + \bar{\zeta}_i\bar{\zeta}_j z_{ij}^2}{|z_{ij}|^5}\right) \tag{10}$$

and the local field in the arbitrary point $\xi \in C$

$$h_\xi = \frac{1}{2}\sum_j\Big(\frac{\zeta_j}{|z_j - \xi|^3} + 3\frac{\bar{\zeta}_j(z_j - \xi)^2}{|z_j - \xi|^5}\Big), \tag{11}$$

where on a complex plane $C$ in $z_j \in C$ there exist point dipole sources $\zeta_j$ . We introduce the torus

$$T^N = \{\zeta = (\zeta_1, ..., \zeta_N) \ : \ |\zeta_i| = 1\} \tag{12}$$

From (12) it follows that $\bar{\zeta}_i = 1/\zeta_i$. Then for the energy of a plane dipole we obtain the following Loran series

$$H = -\frac{1}{8}\sum_{i,j}\left(\frac{\frac{\zeta_i}{\zeta_j} + \frac{\zeta_i}{\zeta_j}}{|r_{ij}|^3} + 3\frac{\zeta_i\zeta_j\bar{z}_{ij}^2 + \frac{z_{ij}^2}{\zeta_i\zeta_j}}{|z_{ij}|^5}\right) \tag{13}$$

The calculation of the algebraic equations

$$\{\partial H/\partial\zeta_i\} = 0 \ \text{ under } |\zeta_i| = 1, \ i = 1, 2, ..., N \tag{14}$$

correspond to an arbitrary energy extrtemum (13) in $C^N$. Equations (14) are also the necessary conditions of the minimum of dipole energy $H$ in local coordinates $\exp(i\vartheta_i)$, where $\vartheta_i$ is the angle characterizing two-dimensional vector $\zeta_i$ on the torus (12) and it takes the form of a system

$$\{\partial H/\partial\vartheta_i\} = 0 \tag{15}$$

Using (8) and (9) and expressing the dipole and the local field (11) on a site $i$ in the form

$$\zeta_i = \exp(i\vartheta_i); \ h_i = |h_i|\exp(i\eta_i) \tag{16}$$

We obtain clearly pronounced expression for (15)

$$\partial H/\partial\vartheta_i = |h_i|\sin(\eta_i + \vartheta_i)/2 = 0 \tag{17}$$

Substituting the solution of the system (17) in (8) written in variables (16)

$$H = -\frac{1}{4}\sum_i(h_i\bar{\zeta}_j + \bar{h}_i\zeta_j) = -\frac{1}{2}\sum_i|h_i|\cos(\eta_i + \vartheta_i) \tag{18}$$

we see that the ground and stable states appear at $\eta_i + \vartheta_i = 0$, i.e. *when the dipole matches local field at a site*. Notice, that the ground state corresponds to the highest possible values of $|h_i|$.

Let us consider plane lattice with a cell in the form of a parellelogram. In each lattice site we put a dipole. The lattice site defined by elementary parallelogram with sides $l$ and $r$ having corner $\phi$. We consider here and in n.2.2 the local field (11) without coefficient 1/2, therefore the problem on stability statescan be formulated as following: for the function

$$h(w_\beta) = \sum_{\alpha\neq\beta}\Big(\frac{\zeta_\alpha}{|w_\alpha - w_\beta|^3} + 3\frac{\overline{\zeta}_\alpha(w_\alpha - w_\beta)^2}{|w_\alpha - w_\beta|^5}\Big), \tag{19}$$

we must find all $\{\zeta_\alpha\}$ (with a) which satisfies the condition

$$\forall\beta\ :\ h(w_\beta) = \lambda_\beta\zeta_\beta,\ \lambda_\beta \in \mathcal{R},\ \mathrm{sgn}\lambda_\beta = const \neq 0. \tag{20}$$

where $w_\alpha$ denoted the variable point on lattice, $w_\beta$ is the fixed point on lattice. and $|\zeta_\alpha|$ for all $\alpha = (m, n)$ (here $(m, n)$ defined the side of point at lattice).

Note that the energy of plane dipole system can be define by following formula

$$H = -\frac{1}{4}\lim_{p\to\infty}[p^{-1}\ \mathrm{Re}\sum_{\beta\in\mathcal{K}_p}\overline{\zeta}_\beta h_p(w_\beta)]. \tag{21}$$

where $\mathcal{K}_p$ denoted the parallelogram with sides parallel to the elementary vector of lattice that contains $p$ lattice points. $h_p(w = \beta)$ is the field induced on the site $w_\beta$ by dipoles found inside $\mathcal{K}_p$. Taking into account the uniformly convergences of the series (19) on $w_\beta$, one can obtain that

$$H = -\frac{1}{4}\lim_{p\to\infty}[p^{-1}\ \mathrm{Re}\sum_{\beta\in\mathcal{K}_p}\overline{\zeta}_\beta h(w_\beta)]. \tag{22}$$

Formula (22) has a simple form particularly if a two-dimensional consequence $\{\lambda_\beta\} = \{\lambda_{MN}\}$ has finite limit when M and N independently tend to infinity. Since $h(w_\beta)\overline{\zeta}_\beta = \lambda_\beta$ see (20) we obtain by the regularity of Chezaro method following

**Proposition 1.** *Let* $\lim_{M,N\to\infty}\lambda_{MN} = \lambda$, *then*

$$H = -\frac{1}{4}\lambda \tag{23}$$

## 2.2 Homogeneous periodic configurations

In this point it will be assumed that $\lambda_{MN} \equiv \lambda = const$ *(homogeneous condition)*. Stabile configurations is being searched in the class of double periodic functions $\zeta_\alpha \equiv \zeta_{mn} = x(m,n) + iy(m,n)$. So let

$$x(m_1,n_1) - x(m_2,n_2) = y(m_1,n_1) - y(m_2,n_2) = 0 \tag{24}$$

if only $m_1 - m_2 \equiv n_1 - n_2 \equiv 0 \ (mod\ s)$, $s$ is fixed natural number. Thus, the condition (20) transforms into the system of lineal equations in the form

$$\begin{array}{l} \sum_{p,q=0}^{s-1}(a_{pq}x(p,q) + b_{pq}y(p,q)) = \lambda x(M,N), \\ \sum_{p,q=0}^{s-1}(b_{pq}x(p,q) + d_{pq}y(p,q)) = \lambda y(M,N), \end{array} \tag{25}$$

where $M,N = \overline{0,s-1}$, $f(p,q) \equiv f(s,M,p,N,q) = f(\{\frac{M+p}{s}\}s, \{\frac{N+q}{s}\}s)$, and $\{\bullet\}$ denotes a fractional part of the number $(\bullet)$,

$$a_{pq} = \sum_{m=p,n=q(mod s)}' \frac{4l^2m^2 + 8lrmn\cos\phi + r^2n^2(1+3\cos 2\phi)}{(l^2m^2 + 2lrmn\cos\phi + r^2n^2)^{5/2}}, \tag{26}$$

$$b_{pq} = \sum_{m=p,n=q(mod s)}' \frac{6(rn\cos\phi + lm)rn\sin\phi}{(l^2m^2 + 2lrmn\cos\phi + r^2n^2)^{5/2}}, \tag{27}$$

$$d_{pq} = \sum_{m=p,n=q(mod s)}' \frac{-2l^2m^2 - 4lrmn\cos\phi + r^2n^2(1-3\cos 2\phi)}{(l^2m^2 + 2lrmn\cos\phi + r^2n^2)^{5/2}}, \tag{28}$$

with normed conditions

$$x^2(M,N) + y^2(M,N) = 1, \text{ where } M,N = \overline{0,s-1}. \tag{29}$$

If digress from the concrete form of the coefficients (26)-(28), lets note that the matrix of system (25) fall into four cells

$$G = \begin{pmatrix} A & B \\ B & D \end{pmatrix} \tag{30}$$

where A, B, D are the block matrix of a specific form, that we call block *circulants*. These block circulants has the form

$$\begin{pmatrix} A_0 & A_1 & \dots & A_{s-1} \\ A_{s-1} & A_0 & \dots & A_{s-2} \\ \dots & \dots & \dots & \dots \\ A_1 & A_2 & \dots & A_0 \end{pmatrix} \tag{31}$$

where $A_0, A_1, \dots, A_{s-1}$ are circulants composed out of elements $a_{00}, \dots, a_{0,s-1}; \dots;$ $a_{s-1,0}, \dots, a_{s-1,s-1}$ respectively. We call to mind that the circulant of the elements $a_0, a_1, \dots, a_{s-1} \in C$ is a square matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_{s-1} \\ a_{s-1} & a_0 & a_1 & \dots & a_{s-2} \\ \dots & \dots & \dots & \dots & \dots \\ a_1 & a_2 & a_3 & \dots & a_0 \end{pmatrix}. \tag{32}$$

We formulate two statements about block circulants eigenvalues proved by I.V.Ermilov.

**Proposition 2.** *For the block circulantA*

$$\det A = \Pi_{j,k=0}^{s-1} \sum_{p,q=0}^{s-1} \epsilon_j^p \epsilon_k^q a_{pq}, \tag{33}$$

*Eigenvalues and corresponding eigenvectors for $A$ are evaluated by the formulas*

$$\lambda_{jk}^A = \sum_{p,q=0}^{s-1} \epsilon_j^p \epsilon_k^q a_{pq}, \tag{34}$$

$$\vec{Z}_{jk} = (1, \epsilon_k, \epsilon_k^2, \dots, \epsilon_k^{s-1}, \epsilon_j, \epsilon_j\epsilon_k, \dots, \epsilon_j\epsilon_k^{s-1}, \dots, \epsilon_j^\alpha \epsilon_k^\beta, \dots, \epsilon_j^{s-1}\epsilon_k^{s-1}), \tag{35}$$

*where $\epsilon_j^\alpha \epsilon_k^\beta$ is $(\alpha s + \beta)$ vector component, $j, k = \overline{0, s-1}$ , and $\epsilon_k$ are roots of $s$ power from unit.*

Using commutational property for the product of one size block circulants and Shur's formula [17] $\det \begin{pmatrix} A & B \\ B & D \end{pmatrix} = \det(AD - ACA^{-1}B)$, where $A, B, C$, and $D$ are block circulants we obtain formulas for eigenvalues of the matrix $G = \begin{pmatrix} A & B \\ B & D \end{pmatrix}$ :

$$\begin{aligned} \lambda'_{jk} &= \tfrac{1}{2}(\lambda_{jk}^A + \lambda_{jk}^D + \sqrt{(\lambda_{jk}^A - \lambda_{jk}^D)^2 + 4\lambda_{jk}^C \lambda_{jk}^B}), \\ \lambda''_{jk} &= \tfrac{1}{2}(\lambda_{jk}^A + \lambda_{jk}^D - \sqrt{(\lambda_{jk}^A - \lambda_{jk}^D)^2 + 4\lambda_{jk}^C \lambda_{jk}^B}), \end{aligned} \tag{36}$$

where $k, j = \overline{0, s-1}$. We remind that if $B = C$ and $a_{pq}, b_{pq}, d_{pq}$ are calculated with formulas (26)-(28), then $G$ is the matrix of a system (25), that we call plane dipole system stable states matrix.

**Theorem 1.** (about eigenvalues of plane dipole system stable states matrix) *Eigenvalues of stable states matrix are determined by formulas (36), with $B = C$, and*

$$\lambda_{jk}^{A} = \sum{}' \frac{4l^2m^2+8lrmn\cos\phi+r^2n^2(1+3\cos 2\phi)}{(l^2m^2+2lrmn\cos\phi+r^2n^2)^{5/2}} \cos\frac{2\pi(jm+kn)}{s},$$
$$\lambda_{jk}^{B} = \sum{}' \frac{6(rn\cos\phi+lm)rn\sin\phi}{(l^2m^2+2lrmn\cos\phi+r^2n^2)^{5/2}} \cos\frac{2\pi(jm+kn)}{s}, \tag{37}$$
$$\lambda_{jk}^{D} = \sum{}' \frac{-2l^2m^2-4lrmn\cos\phi+r^2n^2(1-3\cos 2\phi)}{(l^2m^2+2lrmn\cos\phi+r^2n^2)^{5/2}} \cos\frac{2\pi(jm+kn)}{s}.$$

Lets point out the construction rules of eigenvectors of plane dipole system stable states matrix by its eigenvalues. If the eigenvalue $\lambda$ of $G$ matrix correspond to eigenvector $.\vec{a}$, then we write $\lambda \to .\vec{a}$. Under $(.\vec{a}, \vec{b})$ we mean the vector obtained by adding to vector $.\vec{a}$ the coordinates of vector $\vec{b}$.

**Theorem 2.** (about eigenvectors of plane periodic system stable states matrix). *Eigenvectors of matrix (30) with elements (26)-(28) that correspond to eigenvalues (36a), (36b) are evaluated according to the following four rules:*

*a) if $\lambda_{jk}^{B} \neq 0$, then*

$$\lambda_{jk}' \to (\lambda_{jk}^{B}\vec{Z}_{jk}, (\lambda_{jk}' - \lambda_{jk}^{A})\vec{Z}_{jk}), \quad \lambda_{jk}'' \to (\lambda_{jk}^{B}\vec{Z}_{jk}, (\lambda_{jk}'' - \lambda_{jk}^{A})\vec{Z}_{jk}); \tag{38}$$

*b) if $\lambda_{jk}^{B} = 0$ and $\lambda_{jk}^{A} > \lambda_{jk}^{D}$, then*

$$\lambda_{jk}' \to (\vec{Z}_{jk}, \vec{0}), \quad \lambda_{jk}'' \to (\vec{0}, \vec{Z}_{jk}); \tag{39}$$

*c) if $\lambda_{jk}^{B} = 0$ and $\lambda_{jk}^{A} < \lambda_{jk}^{D}$, then*

$$\lambda_{jk}' \to (\vec{0}, \vec{Z}_{jk}), \quad \lambda_{jk}'' \to (\vec{Z}_{jk}, \vec{0}); \tag{40}$$

*d) if $\lambda_{jk}^{B} = 0$ and $\lambda_{jk}^{A} = \lambda_{jk}^{D}$, then to eigenvalue $\lambda_{jk}' = \lambda_{jk}''$ with multiplicity of two correspond two independent eigenvectors*

$$(\vec{0}, \vec{Z}_{jk}), \quad (\vec{Z}_{jk}, \vec{0}); \tag{41}$$

*where $j, k = \overline{0, s-1}$; coordinates of vector $\vec{Z}_{jk}$ are determined by (35); $\vec{0}$ is a zero vector that as $\vec{Z}_{jk}$ both have component $s^2$.*

## 2.3 Illustration of homogeneous periodic stable states in square lattice

In this paragraph we'll show the usage of theorems 1 and 2 for the construction of stable states. This theorems allow us to find only homogeneous $\lambda_{MN} \equiv \lambda \in \mathbf{R}\backslash\{0\})$ periodic distributions of dipoles in the lattice but only in such cases when among solutions of system (30) one manages to separate vectors which coordinates satisfy nonlinear system of equations (29).
Lets consider a square lattice. Assuming in (37) $l = r = 1$, $\phi = \pi/2$, we obtain:

$$\begin{array}{l} \lambda^A_{jk}(s) = \sum' \frac{4m^2-2n^2}{(m^2+n^2)^{5/2}} \cos\frac{2\pi(jm+kn)}{s}, \\ \lambda^B_{jk}(s) = \sum' \frac{6mn}{(m^2+n^2)^{5/2}} \cos\frac{2\pi(jm+kn)}{s}, \\ \lambda^D_{jk}(s) = \sum' \frac{-2m^2+4n^2}{(m^2+n^2)^{5/2}} \cos\frac{2\pi(jm+kn)}{s}. \end{array} \qquad (42)$$

Notice that among double series $\lambda^A_{jk}(s)$ ($j, k = \overline{0, s-1}$, $s$ is fixed) the series $\lambda^A_{00}(s)$ converges slower of all others. Thus, if a chosen summing border for a series provides an error $\epsilon$, then just the same border for all other series $\lambda^A_{jk}(s)$ would provide an error which is deliberately smaller than $\epsilon$. Noticeably that sum $\lambda^A_{00}(s)$ one can express by single series sums

$$\lambda^A_{00}(s) = 4\sum_{n=1}^{\infty}\frac{1}{n^{3/2}} \cdot \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^{3/2}}. \qquad (43)$$

The last equality in this chain represents a special case of a well known Hardy's result [4]. Equality (43) allows us to define the required border of summing "experimentally". We chose such $N$ that

$$\left| \sum_{m,n=-N}^{N} \frac{4m^2-2n^2}{(m^2+n^2)^{5/2}} - 4\sum_{n=1}^{\infty}\frac{1}{n^{3/2}} \cdot \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^{3/2}} \right| < \epsilon$$

Single series sums $\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ and $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^{3/2}}$ are easily calculated approximately with very high degree of accuracy. The error in approximate calculation of sums $\lambda^D_{jk}$ and $\lambda^B_{jk}$ is determined also by series $\lambda^D_{00}$ and $\lambda^B_{00}$. For all this one should bear in mind that $\lambda^D_{00} = \lambda^A_{00}$, $\lambda^B_{00} = 0$ with any $s$. The construction corresponding to them eigenvectors was held on a computer using (37). Table 1 shows the result with corresponding eigenvalues distribution

(36a) and (36b) of matrix (30).

| $\lambda$ *value* | *multiplicity* $s=4$ | *states* | *energy* |
|---|---|---|---|
| 10.198 | $\lambda'_{02}=\lambda'_{20}$ | *ground state* | −2.55 |
| 9.617 | $\lambda'_{01}=\lambda'_{10}=\lambda'_{03}=\lambda'_{30}$ | 1 *exited* | −2.40 |
| 8.996 | $\lambda'_{00}=\lambda''_{00}$ | *ferromagnetic* | −2.25 |
| 3.605 | $\lambda'_{11}=\lambda'_{33}=\lambda'_{13}=\lambda'_{31}$ | 3 *exited* | −0.9 |
| 3.437 | $\lambda'_{12}=\lambda'_{21}=\lambda'_{23}=\lambda'_{32}$ | 4 *exited* | −0.86 |
| −2.646 | $\lambda'_{22}=\lambda''_{22}$ | 5 *exited* | 0.66 |
| −4.267 | $\lambda''_{11}=\lambda''_{33}=\lambda''_{13}=\lambda''_{31}$ | 6 *exited* | 1.07 |
| −6.454 | $\lambda''_{10}=\lambda''_{01}=\lambda''_{30}=\lambda''_{03}$ | 7 *exited* | 1.61 |
| −7.068 | $\lambda''_{12}=\lambda''_{21}=\lambda''_{23}=\lambda''_{32}$ | 8 *exited* | 1.77 |
| −12.068 | $\lambda''_{02}=\lambda''_{20}$ | *max* | 3.02 |

. (44)

Strict formulation of the ground state problem for the case of square lattice in the form of the following hypothesis. We call a ground state the distribution possessing minimal specific energy.

Strict formulation of the problem about ground state for a square lattice in the form of a following hypothesis. We call the distribution possessing minimal specific energy of interaction the ground state According to proposition 1 and theorem 2 the task of finding of the ground state in a class of periodic homogeneous distributions is formulated as follows. We need to find

$$-\frac{1}{4}\cdot\sup_{s\in|N|}\big(\max_{0\le j,k\le s-1}(\lambda(s),\lambda(s))\big), \tag{45}$$

where $\lambda'$ and $\lambda''$ are calculated by (36)-(37).

In this paragraph we give an equivalent formulation of this problem which - to our mind - is more preferable. It is evident that $\lambda_{''}\le\lambda_{'}$. Then in (45) under the sign max one may write only $\lambda_{'}$. Now we make in (37) substitution of variables

$$u=j/s, v=k/s.$$

As $s=1,2,3...,$ and $j,k=\overline{0,s-1}$, new variables $u,v$ pass through the set of all rational numbers from interval $[0,1)$. Such an assumption can't lead to distortion of the solution of (45) by consideration of continuity.

Examine the function

$$\Psi(u,v)=\sum_{m,n=-\infty}^{\infty}|lm+e^{i\phi}|^{-5}\exp(2\pi i(mu+nv)), \tag{46}$$

and differential operator

$$\delta = \frac{1}{2}(l\frac{\partial}{\partial u} + e^{i\phi}rn\frac{\partial}{\partial v}). \quad (47)$$

Analytical calculations show that problem (45) in this notations has the following form

$$H(u,v) = \frac{1}{4\pi^2}(|\delta|^2\Psi - 3|\delta^2\Psi|) \rightarrow \min, \quad (48)$$

where $0 \le u, v < 1$. Formulas (36)-(37) allow us to put rather a lot point on surface $E(u,v)$. We believe that function (48) takes minimal value in points $(0, 1/2)$ and $(1/2, 0)$. Thus, it 2 2 proves that microvortex distribution is the ground state in a class of homogeneous periodic distributions of dipoles on the infinite square lattice.

## 2.4 About stable vortex and ferromagnetic states of two-dimensional rhombus lattices

In this section we investigate one parameter class of rhombic lattices ($r/l = 1$) set by rhombicity angle $\alpha$. Numerical experiment for finite dipole system ($\alpha = 60^0$) shows that the ground state is not a periodic one [...]. Therefore in this section we give up a priori assumption about periodicity of ground and stable states. We chose the starting point of coordinates at the centre of an elementary rhombus, and the axe $x$ stretch along the greater diagonal. Then the lattice with $\mu = (m,n)$, $\nu = (p,q)$ may be written as

$$\begin{gathered} z_\mu = (m - \tfrac{1}{2})\exp(-i\tfrac{\alpha}{2}) + (n - \tfrac{1}{2})\exp(i\tfrac{\alpha}{2}) = |z_\mu|\exp(i\phi_\mu), \\ z_{\mu\nu} = (m-p)\exp(-i\tfrac{\alpha}{2}) + (n-q)\exp(i\tfrac{\alpha}{2}), \\ \phi_\mu = \operatorname{arctg}(\tfrac{m-n}{m+n-1}\operatorname{tg}\tfrac{\alpha}{2}), \ \ \phi_\nu = \operatorname{arctg}(\tfrac{p-q}{p+q-1}\operatorname{tg}\tfrac{\alpha}{2}), \end{gathered} \quad (49)$$

We assume that the lattice (49) in a discus $D_\rho$ with radius $\rho$. Lets write (13) in the form

$$H = \sum_{\mu,\nu \in D_4} (U_{\mu\nu} + \overline{U}_{\mu\nu}). \quad (50)$$

where summation over all $\mu$ and $\nu$ ($\mu \ne \nu$), i.e. over a set $D_4 = D_\rho \times D_\rho \backslash \Delta$ , where $\Delta$ is a diagonal. From (13) and (16) we have

$$U_{\mu\nu} = \frac{1}{8}([e^{i(\phi_\mu - \phi_\nu)}|z_{\mu\nu}|^2 + 3e^{-i(\phi_\mu+\phi_\nu)}z_{\mu\nu}^2]/|z_{\mu\nu}|^5) \quad (51)$$

Consider the case $\alpha = \pi/3$ (hexagonal or triangle lattice). We compare two configurations: ferromagnetic configuration (F) $\vartheta_\mu = 0$ and macrovortex configuration (R) $\vartheta_\mu = \phi_\mu + \pi/2$. Substituting (51) in (50) and Reducing simultaneously quadratic forms of the numerator and denominator to the main axes using the substitute

$$h = (m - p + n - q)/2, \quad k = (-m + p + n - q)/2, \tag{52}$$

the series is reduced to the form (at $\rho \to \infty$ )

$$\begin{aligned} \lim_{N\to\infty} \frac{H^F}{N} = U^F = -2\sum_{h=1}^{\infty}\sum_{k=1}^{\infty} \frac{6h^2-k^2}{(3h^2+k^2)^{5/2}}, \\ \lim_{N\to\infty} \frac{H^R}{N} = U^R = -2\sum_{h=1}^{\infty}\sum_{k=1}^{\infty} \frac{ah^2+bk^2+chk}{(3h^2+k^2)^{5/2}}. \end{aligned} \tag{53}$$

where

$$a = -\tfrac{1}{2}[\cos(\phi_\mu - \phi_\nu) - 3\cos(\phi_\mu + \phi_\nu)],$$
$$b = -\tfrac{1}{2}[\cos(\phi_\mu - \phi_\nu) + 3\cos(\phi_\mu + \phi_\nu)],$$
$$c = -3\sqrt{3}\sin(\phi_\mu + \phi_\nu).$$

While summing over the lattice of dipoles in the finite disk $D_\rho$ one should keep in mind that meaning of initial sums is preserved, if $\mu, \nu \in D_\rho$. The expressions (53) absolutely converge and one can hope to calculate them analytically. But even in (53b) there appear effects of gigantic oscillations. That leads to the necessity of introducing periodic border conditions as well as using numeric calculations.

## 2.5 Some considerations about local fields structure, connected with limit transition

We try to construct the general picture of possible configurations of ground and stable states of two-dimensional dipole lattices with Hamiltonian (13). To do this we fulfil quality analyses of configuration geometry and use the results of numeric calculations.
This happened due to traditional search of the ground state configuration in the class of periodic solutions, which stability were guaranteed by specification of periodic border conditions. When the latter were lifted (ruled out) (free border), the macrovortex configuration appeared to be the ground

state, and the ferromagnetic configurations were unstable relatively to long-wave fluctuations and had approximate continuous degeneracy.
In this case of finite disks it "seems" to dipoles that the centre as well as the border are limiting ones for the lattice. It automatically produces the vortex. Namely, if a homological border is limiting for the lattice, then dipoles try to arrange parallel with the border. For a finite disk a homological border is a circle. A removed point is also the homological border, and if it is a limiting one for the lattice, then arrows form a vortex around it and are parallel to it.
On the other hand, plane system of dipoles with finite number can't have continuous degeneracy in the ground state. As the degeneracy space of ferromagnetic configuration is a circle - and the degeneracy space of macrovortex consists of two points - then ferromagnetic dipole structure of the ground state for finite disks is impossible. This corresponds with numeric results of the experiment in [19] about unstable ferromagnetic state of a hexagonal lattice in a disk and its disintegration into even number of vortices. Probably the positive difference $\Delta = U^F - U^R$ is connected with lifting of the degeneracy.
In a complex task connected with the determining of the ground state (determining of the global energy minimum) in a three-dimensional system of the field, created by classic dipoles on the infinite two-dimensional lattice, the dependence of the ground state on the nature of the limiting transition may appear. One should consider independence of the limiting transition when $N \to \infty$ and $V \to \infty$, and when the density ($W = N/V$) remains constant. we obtain below the classification parameter which sets up the way of compactification in the dipole two-dimensional lattice in the space $\mathbf{E}^3$ . It's introduction allow us to define that the type of the ground state determines border conditions in the point $\infty$, i.e. global topological properties of a system.
Let us consider several simplest compactifications $\mathbf{E}^2$ embedded into $\mathbf{E}^3$ and $\mathbf{E}^4$ . These are spheric $\mathbf{S}^2$ , toric $\mathbf{T}^2$ and projective $\mathbf{RP}^2$ compactifications. The basis of each compactification is a disk with radius $\rho$, where a lattice with classical point dipoles of unit length. One can easily imagine this parameter while comparing two dipole configurations (R) and (F) on the trigonal lattice (53). It is convenient due to a widely used method of triangulation of compacts. Asymptotic behavior of (53) is the main problem which extrapolation is nontrivial due to bad convergence of $U^F$ and $U^R$ to each other. More-

over, computers can't calculate the energy of systems with great amount of particles interacting all with all according to the law (13). That's making analytical calculations from section 2 one should pay more attention to the conversion into (53).

Notice, that first we made the summation with four-dimensional lattice $\mu, \nu \in D_\rho \times D_\rho \backslash \Delta$. Then we made nonobvious transition by substitution $u = m - p$, $v = n - q$. In this case (53a) and (53b) produced the figure which does not depend on $N$. we managed to obtain the independence of N and go from total energy to the energy per particle, i.e. to the density of energy, in the following way. For ferromagnetic configuration (F) the equality $\lim_{N \to \infty} H^F(N)/N = \sum_p \sum_q H$ is the consequence of Cezare method regularity, if this configuration is on one site ($\mu = const$, $m = 0$, $n = 0$). In the common case the transition

$$\sum_m \sum_n \sum_p \sum_q \psi(m-n, p-q) \to \sum_u \sum_v Q\psi(u, v)$$

, is connected with nonobvious assumption $Q \approx N$ with $N \to \infty$, where $Q$ is the number of solutions of Diophantine system of equations $u = m - n$, $v = p - q$, lying in the area $\mathcal{D}_\rho$. Problems of finding $Q$ in the area include special case of Gauss ($m + n \leq R^2$) and Dirichle ($m \cdot n \leq R^2$) problems. The parameter of compactification is hidden in this very transition. And this vague situation make us assert the possible dependence of the ground state on the way of compactification. Therefore, the main parameter is the global lattice structure or conditions in infinitely remote point.

One may think that the configuration $\mathcal{R}$ will be ground, if the field in sites $z_\alpha \to \zeta_\infty$ will extrapolate to the smooth field $\Phi$ in some compactification. This assumption allows us to make conclusions of the global minimum existence on the formula

$$\sum_{a \in \Phi^{-1}(0)} \mathrm{ind}_a \Phi = \chi, \tag{54}$$

where $\chi$ is Euler characteristic, being the compactification invariant:

$$\chi(\mathbf{S}^2) = 2, \quad \chi(\mathbf{T}^2) = 0, \quad \chi(\mathbf{RP}^2) = 0.$$

We can take the following basis for the proof of microvortex existence by the method of complex variable function [23].

**Theorem** *Any elliptical n-order function $f(z)$ with zeros $\alpha_1, \alpha_2, ..., \alpha_n$ and* *poles $\beta_1, \beta_2, ..., \beta_n$ in the parallelogram of periods (every period is counted according to its multiplicity) can be expressed by Weierstrasse $\sigma$ function :*

$$f(z) = C\frac{\sigma(z-\alpha_1)\sigma(z-\alpha_2)\cdots\sigma(z-\alpha_n)}{\sigma(z-\beta_1)\sigma(z-\beta_2)\cdots\sigma(z-\beta_n)},$$

*where $C$ is constant and*

$$\overline{\alpha}_1 = (\beta_1 + \beta_2 + ... + \beta_n) - (\alpha_1 + \alpha_2 + ... + \alpha_n) = \alpha_1(\mathrm{mod}\ \tau, \tau').$$

This theorem together with "good" structure of the point $\infty$ can allow field $h_\xi$ approximation (11) by some meromorphic field ($\overline{\xi}$ is absent), if in a specified lattice of dipole sources $\beta_1, \beta_2, ..., \beta_n$ a dual lattice of field zeros $\alpha_1, \alpha_2, ..., \alpha_n$ will be formed with $n \to \infty$. So this assumption concludes the analyses of infinite two-dimensional lattices.
From (2.5) it follows that in the ground state the finite system of dipoles - disposed at vertices of a regular $N$-gon - forms the vortex with spin orientations directed along the tangent of the described circle. The demand for polygon regularity is important. For example, even for three dipoles that do not build a regular triangle the above assumption is wrong.
In spite of the fact that the picture of stable dipolic configuration is not so clear we see from the above treatment the tendency of lattice dipole configurations to be antiferromagnetic and vortex in the ground state.

# 3 Three-dimensional dipolic

## 3.1 Degeneracy space and the order parameter

Abstract substance dipolic built from dipoles of the unit length $\vec{P}_i$ - arranged in points $\vec{r}_i \in \mathbf{R}^3$ - and has various properties that imitate real physical systems. Real image of the dipolic is a system of hard spheres with point dipole moments in their centres. The transition to infinite medium is nontrivial in this system. After such transition one can judge about topological properties of the dipolic. Before we build the order parameter, we'll define the degeneracy space. Let's stop on the case of strictly defined ground state of the dipolic with simple cubic lattice. As microscopical discontinuity in orientations is repeated with the period $2a$, we'll glue eight spheres. The result of this procedure depends on the way of gluing. We take the direction of one out of eight dipoles as the degeneracy parameter. To be more exact $.\vec{p}_1 = (\sin\vartheta\cos\phi, \sin\vartheta\sin\phi, \cos\vartheta)$. Probably the sphere $\mathbf{S}^2$ is the degeneracy space of one moment $.\vec{p}_1$. Let's consider Hamiltonian (1) contraction into configurations class with the period

$$\mathcal{H}(\{\vec{P}_j\}) \sim \mathcal{H}(.\vec{p}_1, ..., \vec{p}_8)|_{2a}, \ \vec{P}_j \in \mathbf{S}^2. \tag{55}$$

As vertices of a cube are not marked out we demand the contraction (55) symmetry. It means that

$$\mathcal{H}(.\vec{p}_{j_1}, ..., \vec{p}_{j_8}) = \mathcal{H}(.\vec{p}_1, ..., \vec{p}_8) \tag{56}$$

for any rearrangement $(j_1, ..., j_8)$ of the elements $\{1, ..., 8\}$. After that domain of Hamiltonian definition becomes not Cartesian product of spheres but its symmetrization which is isomorphic to the complex projective space $\mathbf{CP}_8$.

$$\mathrm{\mathbf{Symm}}\ \Pi_{i=1}^{8}\mathbf{S}_i^2 \simeq \mathbf{CP}_8. \tag{57}$$

A set of eight nonlabeled vectors in the point (57) - or which is the same, a set $\mathbf{CP}_8$ of polynomial roots of the 8th order - is compact and free from affinization. The sphere $\mathbf{S}_2$ in $\mathbf{CP}_8$ is a nontrivial two-dimensional homological class, i.e. it can't be contracted to a point remaining in space $\mathbf{CP}_8$ . Notice, that the degeneracy space in the set of configurations $(\mathbf{S}_1^2 \times \ ... \ \times \mathbf{S}_8^2)$ before symmetrization is also a nontrivial two-dimensional homological class. Obviously (57) is the basis for the transition from a lattice dipolic to continuous medium. Not all configurations in $\mathbf{CP}_8$ are equally probable due to dipole interaction. For a spherical dipolic having a simple cubic lattice $g = (h, k, l)$ the ground state can be written as [18]

$$\eta^{-1} : \left( \begin{array}{c} P^x_{hkl} = (-1)^{k+l}\sin\vartheta\cos\phi \\ P^y_{hkl} = (-1)^{h+l}\sin\vartheta\sin\phi \\ P^z_{hkl} = (-1)^{h+k}\cos\vartheta \end{array} \right) \tag{58}$$

Foursublattice microvortex structure of the ground state (58) specifies a tetrahedron $\mathcal{T}$ on a set of $\mathbf{CP}_8$ configurations. Denoting

$$a = \sin\vartheta\cos\phi, \quad b = \sin\vartheta\sin\phi, \quad c = \cos\vartheta, \tag{59}$$

we have

$$\mathcal{T}:\left(\begin{array}{c}\vec{P}_1=\vec{P}_{000}=(a,b,c)=\vec{P}_5=\vec{P}_{111}\\ \vec{P}_2=\vec{P}_{100}=(a,-b,c)=\vec{P}_6=\vec{P}_{011}\\ \vec{P}_3=\vec{P}_{110}=(-a,-b,c)=\vec{P}_7=\vec{P}_{001}\\ \vec{P}_4=\vec{P}_{010}=(-a,b,-c)=\vec{P}_8=\vec{P}_{101}\end{array}\right) \tag{60}$$

where $a^2+b^2+c^2=1$.

A regular tetrahedron (60) fully presents dipole order of the ground state and has outstanding features. The parallelepiped $\Pi$, in which the tetrahedron is inscribed, crosses coordinate axes in the way whereas $x$, $y$ and $z$ run through faces centres and parallel to the ribs of $\Pi$. Tetrahedron ribs are diagonals of faces of the tetrahedron inscribed in the sphere with $\mathbf{S}^2$ border. The centres $\Pi$, $\mathbf{B}^3$ and $\mathcal{T}$ coincide. Two–parameter tetrahedron family $\{\mathcal{T}\}$ is a subset $\mathbf{CP}_4 \subset \mathbf{CP}_8$. The tetrahedron $\mathcal{T}$ is an order parameter and its dependence on the degeneracy parameter is defined by (60). Long range order in the above system exists only at the temperature $T=0$, whereas at any $T>0$ long range order is destroyed according to Bor-Van Leven theorem [24]. This order parameter might be useful in the analyses of phases, for instance, Berezinsky phase [13, 32], when there is certain behavior of correlation functions in the absence of long range order. We think it necessary to make mathematical analyses of the order parameter as the way of correlation destruction in the transition from ground to Gibb's states one should investigate using metastables with the period $2a$ and distribution functions on a set of tetrahedrons $\{\mathcal{T}\}$. It is clear that the ground state destruction is connected with several mechanisms such as tetrahedron abrupt reduction in antiferromagnetic, nematic states; tetrahedron separation from crystallographic axes; tetrahedron distortions up to isotropic paramagnetic state, when any configuration on $\mathbf{CP}_8$ is equiprobable.

Let us investigate first a set of symmetrized configurations on $\mathbf{CP}_4$ . We examine the ground state, keeping in mind that its destruction will be connected with metastables with a period $2a$ having the lowest energy. Let's use stereographic projection

$$\frac{a}{1-c}+i\frac{b}{1-c}=z. \tag{61}$$

For the transition from a complex variable $z$ to (59) we use the following formulas

$$a = \frac{2(z+\bar{z})}{1+|z|^2},\ b = \frac{2(z-\bar{z})}{1+|z|^2},\ c = \frac{|z|^2-1}{1+|z|^2}. \tag{62}$$

Using (61) and tetrahedron properties (60) we introduce variables

$$\begin{array}{ll} z_1 = \frac{a}{1-c} + i\frac{b}{1-c}, & z_2 = \frac{a}{1+c} - i\frac{b}{1+c}, \\ z_3 = \frac{-a}{1-c} - i\frac{b}{1-c}, & z_4 = \frac{-a}{1+c} + i\frac{b}{1+c}. \end{array} \tag{63}$$

It is easily seen that

$$z_1 = t,\ z_2 = \bar{t}|t|^2,\ z_3 = -t,\ z_4 = -\bar{t}|t|^2.(4.10) \tag{64}$$

We introduce new variables by means of Viet theorem

$$\begin{gathered} \sigma_1 = -(z_1 + z_2 + z_3 + z_4), \\ \sigma_2 = z_1z_2 + z_1z_3 + z_1z_4 + z_2z_3 + z_2z_4 + z_3z_4, \\ \sigma_3 = (z_1z_2z_3 + z_1z_3z_4 + z_2z_3z_4), \\ \sigma_4 = z_1z_2z_3z_4. \end{gathered} \tag{65}$$

As

$$\Pi_{j=1}^{4}(z - z_j) = z^4 + \sigma_1 z^3 + \sigma_2 z^2 + \sigma_3 z + \sigma_4, \tag{66}$$

then we use (64) and have

$$\sigma_1 = 0,\ \sigma_2 = -t^2(1+\bar{t}^4),\ \sigma_3 = 0,\ \sigma_4 = t^4\bar{t}^4. \tag{67}$$

Coordinatization of $\mathbf{CP}_4$ by means of (67) let us see the nature of low energy metastables with a period $2a$. Due to (67) a set of the ground state configurations are compared with a set of all polynomials

$$\{z\ [z^4 - t^2(1+\bar{t}^4)z^2 + t^4\bar{t}^4]\}, \tag{68}$$

taken as "zero" reading on the set of all polynomials (66). One can see true analogy of (68) with free energy expansion in a series according to the order parameter and which specifies the form of a functional. It is obvious expression is obtained by substituting (59) and (61) in (68).
Different conditions of connection of sublattices with each other and with coordinate axes would give different functions on $\mathbf{CP}_8$ and therefore would specify correlation function decay. Dipolar local order parameter $\mathcal{T}$ may be specified as tensor field $\mathcal{T}(\S)$ in the spirit of the crystal ordering theory.

## 3.2 Topological metastable structures without disclinations

We set the family of pairwise hooking curves along which the tetrahedron (60) is preserved. For this reason we consider homotopic nontrivial mapping $\mathbf{R}^3 \subset \mathbf{S}^3 \to \mathbf{S}^2$. This mapping is Hopf mapping.
Let $(\xi_1 + i\xi_2, \xi_3 + i\xi_4)$ is the coordinates of the complex space $C$, $a + ib \in \mathbf{R} \subset \mathbf{S}^2$ - the point in $\mathbf{S}^2$, that characterizes a complex straight line $C$ running through 0.
The condition

$$\xi_1 + i\xi_2 = (a + ib)(\xi_3 + \xi_4) \tag{69}$$

is equivalent to

$$\begin{aligned} \xi_1 &= a\xi_3 - b\xi_4 \\ \xi_2 &= b\xi_3 + a\xi_4. \end{aligned} \tag{70}$$

Let

$$\xi_1 = \frac{2x}{1+|r|^2},\ \xi_2 = \frac{2y}{1+|r|^2},\ \xi_3 = \frac{2z}{1+|r|^2},\ \xi_4 = \frac{1-|r|^2}{1+|r|^2}\ : \tag{71}$$

is the mapping opposite to stereographic projection, i.e. it fulfils $\mathbf{R}^3 \subset \mathbf{S}^3 \subset \mathbf{R}^4 \simeq \mathbf{C}^2$ embedding. Then a circle resulting from section of the sphere $\mathbf{S}^3 = \{\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 = 1\}$ by the straight line (70) will turn after stereographic projection into a curve

$$\begin{aligned} \tfrac{2x}{r^2+1} &= a\tfrac{2z}{r^2+1} - b\tfrac{1-r^2}{r^2+1} \\ \tfrac{2y}{r^2+1} &= b\tfrac{2z}{r^2+1} + a\tfrac{1-r^2}{r^2+1} \end{aligned} \tag{72}$$

or with $|r|^2 = x^2 + y^2 + z^2$ :

$$\begin{aligned} 2(x - az) &= -b(1 - r^2) \\ 2(y - bz) &= a(1 - r^2) \end{aligned} \tag{73}$$

This system of equations specifies the family of pairwise hooked curves along which a tetrahedron is the same. Two-parameter family of 2 order closed curves (73) is ellipses without a propeller in the space and they completely fill $\mathbf{R}^3$ everywhere. One of the ellipses that goes through the point $\infty$ is degenerated and becomes a straight line (axis $z$). From (73) we find obvious form of the "magnetization vector" $\vec{m} = (m_1, m_2, m_3)$, namely vector field

$.\vec{m}(x,y,z)$ , which is constant for every level line, specified by the ellipses (73). For this purpose we resolve (73) with respect to $a$ and $b$:

$$
\begin{aligned}
a &= \frac{2[2xz-y(1-r^2)]}{4z^2+(1-r^2)^2} \\
b &= \frac{2[2yz+x(1-r^2)]}{4z^2+(1-r^2)^2}
\end{aligned}
\tag{74}
$$

Now we embed the complex plane $\mathbf{C}$ of variables $a+ib$ by means of stereographic projection into a unit sphere

$$
\mathbf{S}^2 = \{(m_1, m_2, m_3) \in \mathbf{R}^3 \ : \ m_1^2 + m_2^2 + m_3^2 = 1\}.
$$

We have

$$
m_1 = \frac{2a}{1+a^2+b^2}, \ m_2 = \frac{2b}{1+a^2+b^2}, \ m_3 = \frac{a^2+b^2-1}{a^2+b^2+1}. \tag{75}
$$

Representing from (74) in (75) the expressions for $a$ and $b$ through $x, y, z$ we get vector field $.\vec{m}(x,y,z)$ obvious form:

$$
\begin{aligned}
m_1(x,y,z) &= (\tfrac{2}{1+r^2})^2[-y-2xz+yr^2], \\
m_2(x,y,z) &= (\tfrac{2}{1+r^2})^2[x+2yz-xr^2], \\
m_3(x,y,z) &= -1+(\tfrac{2}{1+r^2})^2[2x^2+2y^2],
\end{aligned}
\tag{76}
$$

Radian measure

$$
\mu_r = \frac{2}{1+r^2} \tag{77}
$$

results from $\mathbf{R}^3 \subset \mathbf{S}^3 \subset \mathbf{R}^4 \simeq \mathbf{C}^2$ embedding and effectively determines a topological "charge".

Thus, we built the topological soliton without parameters with Hopf invariant which is equal to one. It can't be transferred into dipolar ground state by means of continuous transformation.

At this point we conclude the discussion about particle like excitations in a dipolic. We realize that Hopf soliton may not provide the minimum to a functional from gradient energy. The obtained soliton may result from an eigenfunction of Schroedinger operator. We hope that nice properties of Hopf soliton fully describe the physics of metastables in the dipolic.

**Conclusions** The article deals with geometrical characteristics of balanced lattice dipolar configurations.

The analyses of the problem of conditionally converging dipolar sums showed that the basic idea in the dipolar ground state theory is in the necessity of introducing the lattice. Otherwise the ground state may not exist. Strictly speaking, using the results obtained on lattice systems in the problem of amorphous dipolics is a nontrivial procedure which demands new theoretical and experimental approaches.

In the analytical study of two-dimensional lattices from three-dimensional dipolar sources the usage of methods of complex variable function theory appeared to be effective and led in some cases to formulas in a closed form. Difficulties of taking into account all with all dipolar interactions do not always allow top fulfil up to the end analytical calculations. A probable statement about impossible continuous degeneracy in finite two-dimensional dipolics in spite of introducing in them the lattice structure. This result is important for the understanding of binary electrical level forming from a discrete system of charges.

The existence of antiferromagnetic state and vortices appeared to be the main tendency. The main reasons of dipolar lattices ground state were classified. However, the problem of disproportional and modulated configurations in oblique angled lattices has not yet solved. The methods developed in the article allow us to analyze this problem in future.

For three-dimensional cubic dipolic we obtained degeneracy space and built microscopical order parameter that coincides with macroscopic order parameter in the ground state. This is the tetrahedron $\mathcal{T}$, constructed of the four nearest dipoles. Using symmetrized product of spheres we built the main construction for the transition from lattice dipolics to continuous medium.

Continuous medium with complex projective space $\mathbf{CP}_3$ in every point let us consider topological metastable structures without disclinations in the dipolic as well as in the ferromagnetic. The obvious form of Hopf three-dimensional soliton was found. It is interesting to notice that it does not provide extreme solution to the gradient functional. We managed to obtain an obvious form of Maxwell equations for the vector field, specified by Hopf soliton.

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